Extremal, axisymmetric horizons in the presence of a cosmological constant

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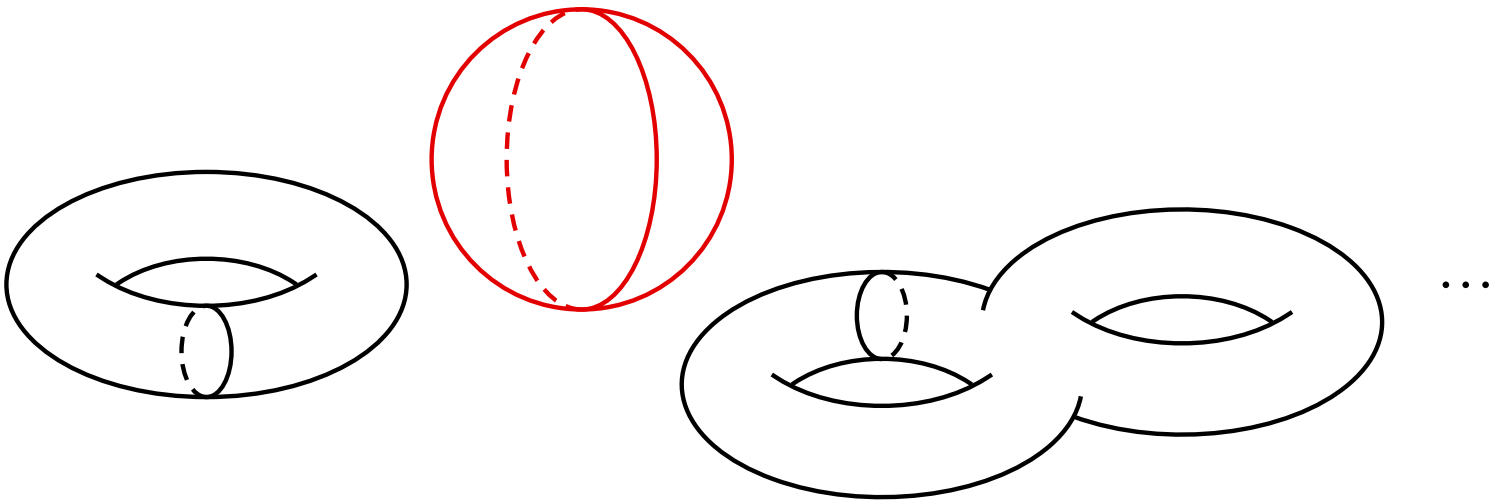
Quasi-local framework



The idea is to relax definition of Killing horizon, while maintaining emough structure to define meaningful physical quantities. It describes horizon of black hole in exact or approximate equilibrium or in isolation.

We will be operating in 4-dimensional spacetime, obeying vacuum Einstein equations:

$$R^{(4)}_{\mu\nu} - \frac{1}{2}R^{(4)}g^{(4)}_{\mu\nu} + \Lambda g^{(4)}_{\mu\nu} = 0. \tag{1}$$

Non-expanding horizon

Let (H, q_{ab}) be smooth, null hypersurface in spacetime, generated by ℓ. It is a null congruence, which means we can define its optical scalars – expansion, shear and twist. We require the first one to vanish.

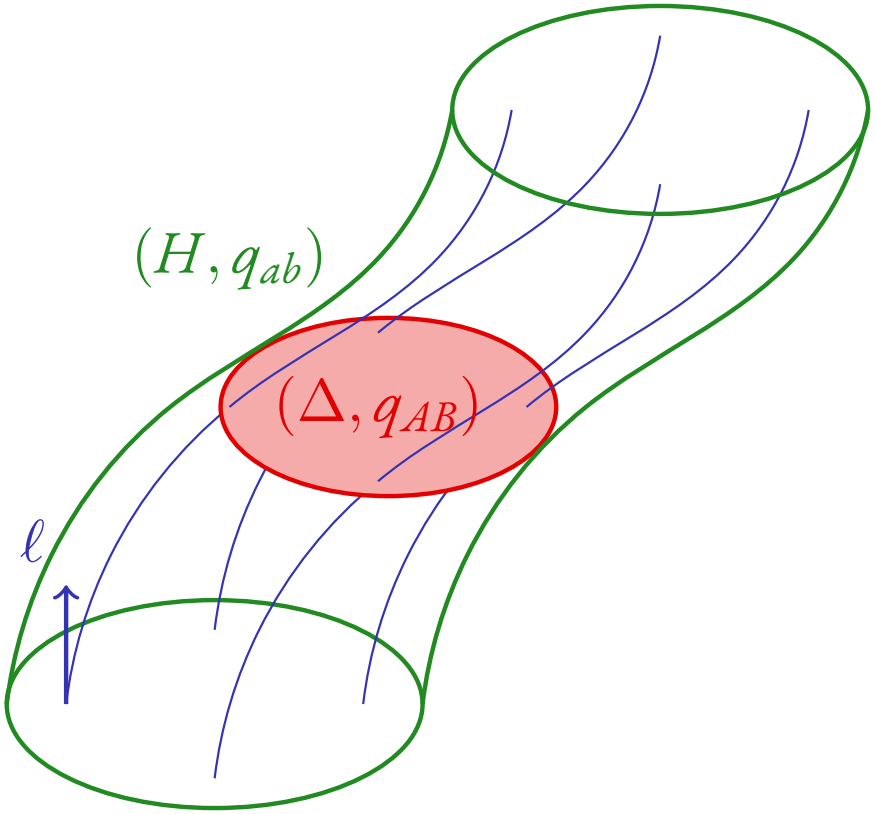
We shall assume, that vector ℓ satisfies dominant energy conditions, which forces other scalars to vanish by Raychaudhuri equation.

Horizon is topological product

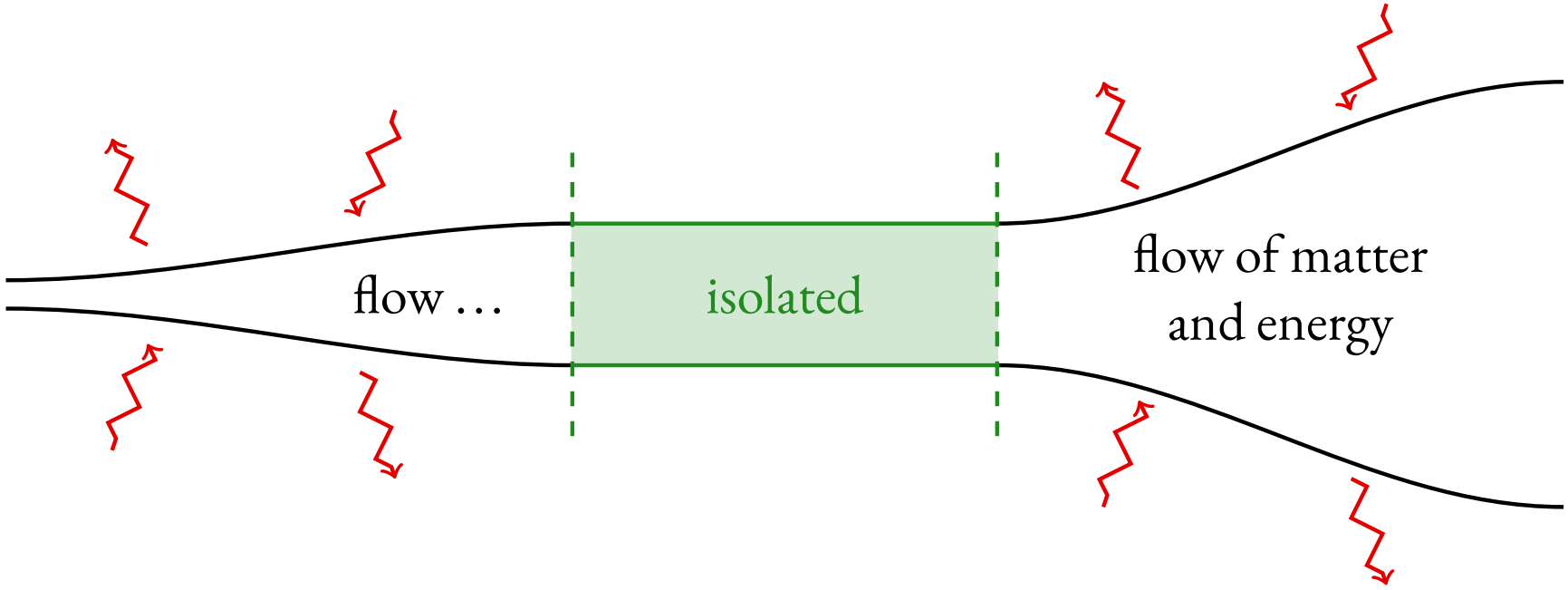
$$H \cong \Delta \times \mathbb{R}, \qquad (2)$$

where section Δ is closed.

Einstein equations are satisfied *on* a horizon.



Isolated horizon



For black holes in equilibrium it should suffice to set geometry (covariant derivative) of a horizon to be *time*-independent:

$$[\mathcal{L}_\ell, \nabla_a]\, V^b = 0, \quad \text{where} \quad V^b \subset T(H). \tag{3}$$

Rotation one-form and NHG equations

Covariant derivative of ℓ

$$\nabla_a \ell^b = \omega_a \ell^b, \tag{4}$$

defines rotation one-form ω, which is tied to angular momentum of black hole. The quantity

$$\kappa^{(\ell)} = \omega_a \ell^a = \nabla_a \ell^a \tag{5}$$

is called surface gravity. When $\kappa^{(\ell)} = 0$ horizon is called extremal (degenerate).

Let Δ be spacelike section of H, transversal to ℓ. Then Einstein equations induce constraints

$$\nabla_{(A}\omega_{B)} + \omega_A \omega_B - \frac{1}{2} R_{AB} + \frac{\Lambda}{2} q_{AB} = 0, \tag{6}$$

called near-horizon geometry (NHG) equations. By integrating this equation, It can be shown, that there must be

$$\Lambda \leq K. \tag{7}$$

Axisymmetry

From here on out, we will be interested in the $\Delta \cong \mathbb{S}_2$

We will be considering axisymmetric Δ, and so we will introduce coordinates (θ, ϕ), such that symmetry is generated by ∂_ϕ. Then the general form of q is

$$q = \Sigma^2(\theta)\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{8}$$

Next we introduce coordinate x such that

$$dx = \frac{\Sigma^2(\theta)\sin\theta}{R^2}d\theta, \quad \text{Area}(\Delta) = 4\pi R^2, \quad x \in [-1, 1]. \tag{9}$$

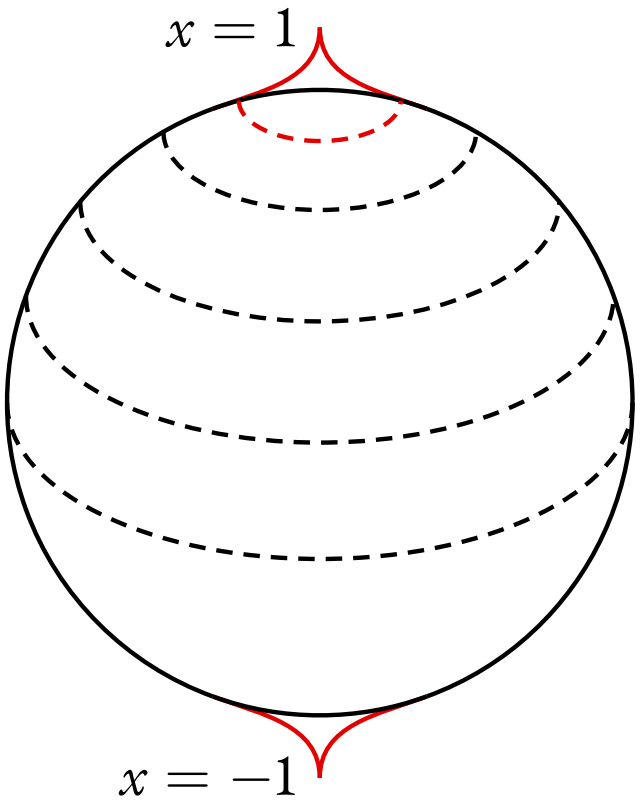
Now metric is given by

$$q = R^2\left(\frac{1}{P^2(x)}dx^2 + P^2(x)d\phi^2\right), \quad P^2(x) = \frac{\Sigma^2(\theta)\sin^2\theta}{R^2}. \tag{10}$$

Regularity conditions and potentials

Function P must admit following limits:

$$P(x = \pm 1) = 0, \quad \partial_x P^2(x = \pm 1) = \mp 2. \tag{11}$$



These conditions follow from regularity of metric and smooth functions on the poles.

We will use Hodge decomposition of rotation one-form

$$\omega = \star dU + d \log B, \tag{12}$$

where potentials U and B are functions of x only.

We end up with 3 differential equations for U, B and function P, which we can solve.

Solution

Solution is given by

$$P^2(x) = \left(x^2 - 1\right) \frac{\Lambda R^2 \left(\Lambda R^2 - x^2(\Lambda R^2 - 1) - 5\right) + 6}{\Lambda R^2 + 3x^2(\Lambda R^2 - 1) - 3}, \tag{13}$$

and parameters are constrained in the following way:

$$\Lambda R^2 \in \left]-\infty, 1\right[\iff \Lambda < \frac{1}{R^2} = K. \tag{14}$$

One-form potentials are given by

$$B^2 = B_0^2 \left[\Omega^2 + x^2\right] \quad \text{and} \quad U = \arctan\left(\frac{x}{\Omega}\right) + U_0, \tag{15}$$

where B_0, U_0 are constant and

$$\Omega^2 = \frac{1 - \frac{1}{3}\Lambda R^2}{1 - \Lambda R^2} \tag{16}$$

For $\Lambda R^2 = 1$ one-form ω vanishes, potentials are constant, and we recover spherical metric:

$$P^2(x) = 1 - x^2, \quad \Lambda = K = \frac{1}{R^2}. \tag{17}$$

Kerr-(anti)de Sitter metric

Kerr-(anti)de Sitter metric has the form

$$g = -\frac{\Delta_r}{\chi^2\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\Delta_\theta \sin^2\theta}{\chi^2\rho^2}\left(adt - \left(r^2+a^2\right)d\phi\right)^2 + \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) \tag{18}$$

where:

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta \\ \Delta_\theta &= 1 + \frac{1}{3}\Lambda a^2\cos^2\theta \\ \chi &= 1 + \frac{1}{3}\Lambda a^2 \\ \Delta_r &= \left(r^2+a^2\right)\left(1 - \frac{1}{3}\Lambda r^2\right) - 2Mr \end{aligned} \tag{19}$$

Zeroes of the last polynomial define horizons (for r,t=const.).

Embedding in Kerr-(anti)de Sitter

By comparing metric of Kerr-(anti)de Sitter's horizon (constant r and t, $\Delta_r(r_0) = 0$) with our form we get:

$$P^2 = (1 - x^2)\frac{1 + \frac{1}{3}\Lambda a^2 x^2}{1 + \frac{1}{3}\Lambda a^2}\frac{r_0^2 + a^2}{r_0^2 + a^2 x^2}, \qquad R^2 = \frac{r_0^2 + a^2}{1 + \frac{1}{3}\Lambda a^2}. \tag{20}$$

Vanishing of determinant of Δ_r (multiple roots) in our parametrization yields:

$$\begin{aligned} a^2 &= \frac{3R^2(1 - \Lambda R^2)}{(\Lambda R^2 - 3)(\Lambda R^2 - 2)} \\ M &= \frac{2}{3}\sqrt{\frac{R^2}{2 - \Lambda R^2}}\frac{(3 - 2\Lambda R^2)^2}{(3 - \Lambda R^2)(2 - \Lambda R^2)} \end{aligned} . \tag{21}$$

Parameters are constrained by

$$\Lambda R^2 \in]-\infty, 1], \tag{22}$$

where limit $\Lambda R^2 = 1$ recovers non-rotating case.

These results agree with earlier ones.

Embedding in Kerr-(anti)de Sitter

Details concerning type and numbers of spacetime horizons are in the table below:

Parameter ranges	Number and type of horizons
$\Lambda R^2 \leq 0$	one extremal, double root
$\Lambda R^2 \in]0,1] \setminus \left\{ \frac{3-\sqrt{3}}{2} \right\}$	one extremal (double root) and one not
$\Lambda R^2 = \frac{3-\sqrt{3}}{2}$	1 extremal, merging of 3 roots

Integrability conditions and type D equation

At an isolated horizon, two principal null directions of Weyl tensor must vanish, therefore it must be of type D, type II or it must be identically zero.

With the assumption, that the Weyl tensor is of type D, we can deduce an integrability condition for **non-extremal** horizon

$$\left(\bar{\delta} + \alpha - \bar{\beta}\right) \bar{\delta} \Psi_2^{-\frac{1}{3}} = 0, \tag{23}$$

where Ψ_2 is Weyl tensor coefficient in Newman-Penrose tetrad. It yields the same metric (same $P(x)$), for $\Lambda R^2 \in]-\infty, 1[$.

Integrability condition has solutions outside of these limits.

Thank You for Your attention.