Axisymmetric, extremal horizons in the presence of a cosmological constant

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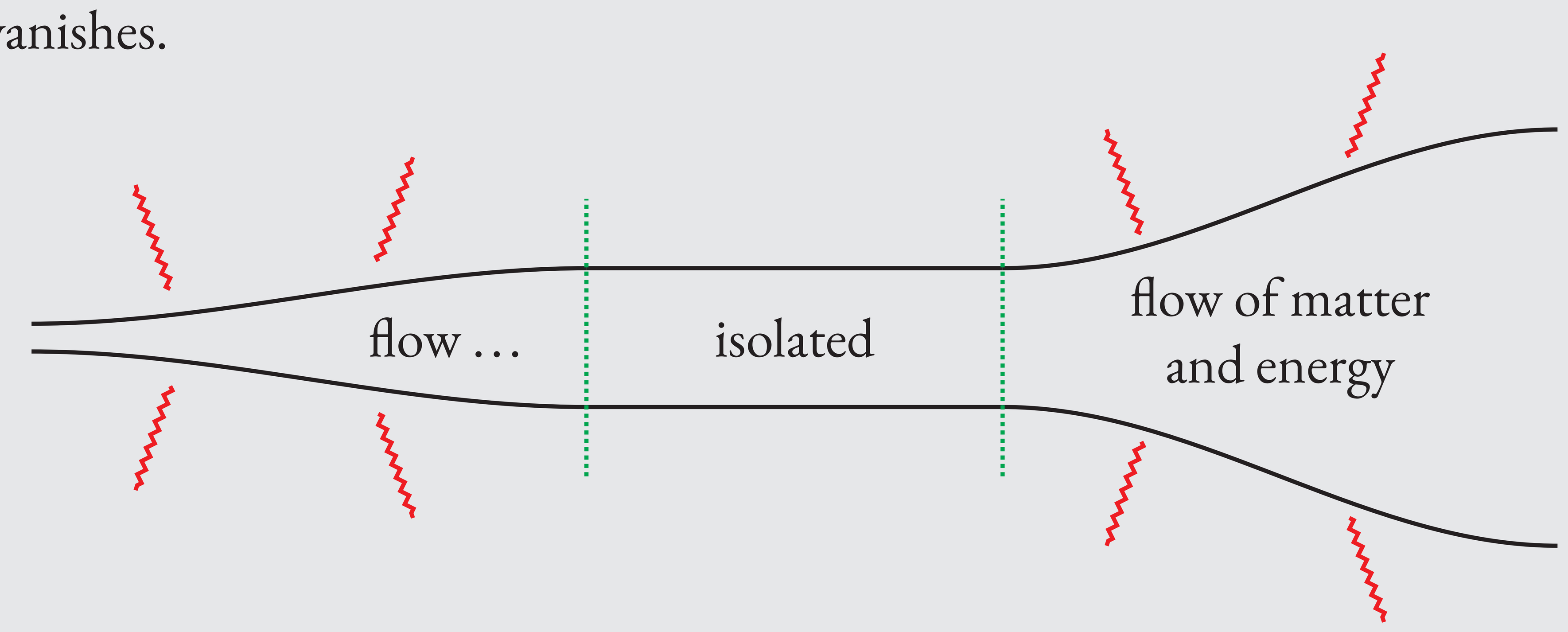
Near-horizon geometry and isolated horizons

Non-expanding horizon models a black hole, that has constant area. It is a null hypersurface generated by nowhere-vanishing vector ℓ, with degenerate metric tensor q and covariant derivative D induced by its space-time counterpart. In particular it can be Killing horizon. We assume, that expansion of ℓ vanishes.



Isolated horizon is a non-expanding horizon, whose geometry is *time-independent*, that is

$$[D, \mathcal{L}_\ell] = 0. \tag{1}$$

It models black hole with no mass-energy transfer through its surface. We can define the rotation one-from ω_a on isolated horizon, together with surface gravity κ:

$$D_a \ell^b = \omega_a \ell^b, \qquad \kappa = \omega_a \ell^a. \tag{2}$$

When surface gravity vanishes, the isolated horizon is called **extremal**.

Near-horizon geometry equations

Isolated horizon is topologically a product $\mathbb{R} \times S$, where a spacelike (indices $A,B,C,\dots$) section S is a compact manifold. Einstein vacuum equations with Λ induce the following constraints on geometry of our section:

$$\nabla_{(A}\omega_{B)} + \omega_A\omega_B - \frac{1}{2}R^{(S)}_{AB} + \frac{\Lambda}{2}q_{AB} = 0. \tag{3}$$

We call these the **Near-horizon geometry (NHG) equations**. One can prove, that for genus$(S) \neq 0$ the solution is trivial

$$\omega = 0, \qquad \frac{1}{2}R = \Lambda, \tag{4}$$

therefore we will be interested in the case $S \cong \mathbb{S}_2$.

Solution

Via Hodge decomposition, we can express rotation one-form as

$$\omega = \star dU + d\log B, \tag{5}$$

where scalar functions U and B are unique up to additive and multiplicative constants U_0 and B_0 respectively. It lets us solve NHG equations, yielding

$$\begin{aligned} B^2 &= B_0^2\left[\Omega^2 + x^2\right] \\ U &= \operatorname{arc\,tg}\left(\frac{x}{\Omega}\right) + U_0 \end{aligned} \quad \text{where} \quad \Omega^2 = \frac{1 - \frac{1}{3}\Lambda\rho^2}{1-\Lambda\rho^2} \tag{6}$$

together with

$$P(x) = (x^2-1)\frac{\Lambda\rho^2\left(\Lambda\rho^2 - x^2(\Lambda\rho^2-1) - 5\right) + 6}{\Lambda\rho^2 + 3x^2(\Lambda\rho^2-1) - 3}. \tag{7}$$

Positivity of metric and Ω^2 restricts possible values of cosmological constant to

$$\Lambda\rho^2 \in]-\infty, 1[. \tag{8}$$

Solution for $\Lambda\rho^2 = 1$ corresponds to spherically-symmetric horizon, and is

$$\omega = 0, \quad P^2 = 1 - x^2, \quad \frac{1}{2}R = \Lambda. \tag{9}$$

At an isolated horizon, two principal null directions of Weyl tensor must vanish, therefore it must be of type D, type II or it must be identically zero. With the assumption, that the Weyl tensor is of type D, we can deduce an integrability condition for **non-extremal** horizon, resulting in the same metric, for $\Lambda\rho^2 \in]-\infty, 1[$. Integrability condition has also solutions outside of these limits.

Summary

All axisymmetric solutions to the near-horizon geometry equation with a cosmological constant defined on a topological 2-sphere were derived. The regularity conditions preventing cone singularity at the poles were accounted for. The one-to-one correspondence of the solutions with the extremal horizons in the Kerr-(anti-)de Sitter spacetime was found. A solution corresponding to the triply degenerate horizon was identified and characterized. The solutions were also identified among the solutions to the Petrov type D equation.

Adapted coordinates

The general metric of an axially symmetric, two-dimensional manifold is

$$q = \Sigma^2(x)\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{10}$$

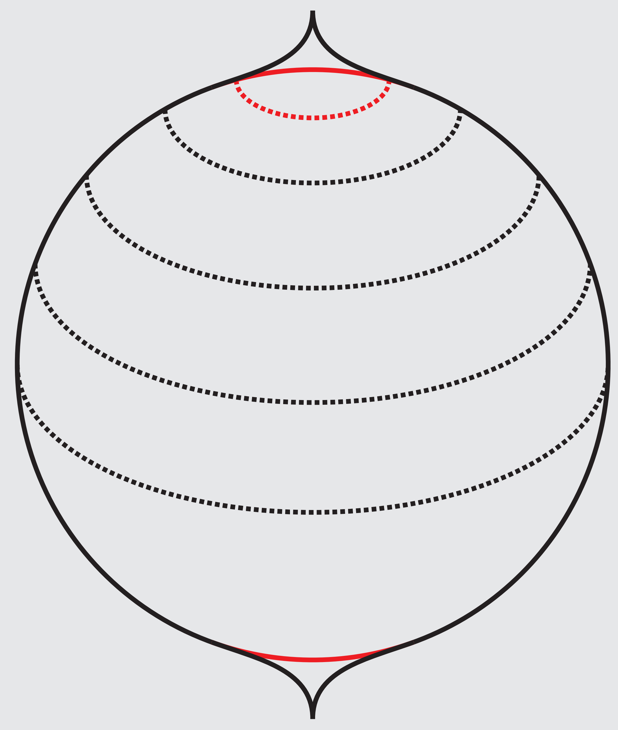
where ∂_φ generates rotational symmetry. We introduce the more convenient coordinate x which is tied to usual θ coordinate by

$$dx = \frac{\Sigma^2(\theta)\sin\theta}{\rho^2}d\theta, \quad x \in [-1,1]. \tag{11}$$

Now metric is of the form

$$q = \rho^2\left(\frac{dx^2}{P^2(x)} + P^2(x)d\varphi^2\right) \quad P^2(x) = \frac{\Sigma^2(x)\sin^2\theta}{\rho^2}, \tag{12}$$

where ρ serves as *radius* of our horizon i.e. Area$(S) = 4\pi\rho$.



It follows from definition of P, that it must vanish for $x = \pm 1$. Moreover to avoid conical singularity, we have ensure that for any δx the circle of such radius around each pole is of length $2\pi\delta x + o(x)$. We have to take $\partial_x(P^2) = \mp 2$ in poles ($x = \pm 1$). Equivalently it is a condition for metric continuity. These boundary conditions for P are crucial to our results.

Embedding in Kerr-(anti-)de Sitter spacetime

Kerr-(anti-)de Sitter spacetime has three horizons (in general) and, aside from Λ, is described by two parameters: M and a – mass of a black hole and its angular momentum (per unit mass). If at least 2 horizons merge, then these parameters can be expressed by

$$\begin{aligned} a^2 &= \frac{3\rho^2(1-\Lambda\rho^2)}{(3-\Lambda\rho^2)(2-\Lambda\rho^2)} \\ M &= \frac{2}{3}\sqrt{\frac{\rho^2}{2-\Lambda\rho^2}}\frac{(3-2\Lambda\rho^2)^2}{(2-\Lambda\rho^2)(3-\Lambda\rho^2)}, \end{aligned} \tag{13}$$

and their positivity forces the same restrictions on the value of cosmological constant, as before. Details concerning type and numbers of spacetime horizons are in the table below:

Parameter ranges	Number and type of horizons
$\Lambda R^2 \neq 0$	one extremal
$\Lambda R^2 \in]0,1] \setminus \left\{\frac{3-\sqrt{3}}{2}\right\}$	one extremal and one not
$\Lambda R^2 = \frac{3-\sqrt{3}}{2}$	merging of all three

These results describe one-to-one correspondence between general axisymmetric extremal horizon and extremal horizon in Kerr-(anti-)de Sitter spacetime.

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