Petrov type D equation on horizons of nontrivial bundle topology

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Introduction

We consider 3-dimensional isolated horizons (IHs) generated by null curves that form nontrivial $U(1)$ bundles. We find a natural interplay between the IH geometry and the $U(1)$-bundle geometry. In this context we derive all axisymmetric solutions to the Petrov type D equation assuming embeddability in 4-dimensional spacetime satisfying Einstein equations with cosmological constant [1].

Isolated horizon structure on $U(1)$ bundle

Let

$$\Pi : H \to S \tag{1}$$

be a principal fiber bundle with the structure group $U(1)$. Denote by ℓ the fundamental vector field on H (dimH=3), such that its flow coincides with the action of $U(1)$ on H. We normalize ℓ so that the parameter of the flow ranges the interval $[0, 2\pi]$. On H we introduce an IH geometry compatible with the bundle structure. It consists of a degenerate metric tensor g_{ab} of the signature $0++$, such that

$$\ell^a g_{ab} = 0 = \mathcal{L}_\ell g_{ab}; \tag{2}$$

and a covariant derivative ∇_a on $T(H)$, torsion free and satisfying:

$$\nabla_a g_{bc} = 0, \qquad [\mathcal{L}_\ell, \nabla_a] = 0. \tag{3}$$

It follows, that

$$\ell^a \nabla_a \ell^b = \kappa \ell^b \tag{4}$$

and we assume that κ is a nonzero constant. The key ingredient of the covariant derivative is the rotation 1-form potential ω_a defined as

$$\nabla_a \ell^b = \omega_a \ell^b. \tag{5}$$

Connection 1-form on the $U(1)$ bundle (1) reads: $\tilde{\omega} := \frac{1}{\kappa}\omega$. The degenerate metric tensor g_{ab} induces on S a (genuine) metric tensor g_{AB} such that g_{ab} is its pullback,

$$g_{ab} = \Pi^*{}_{ab}{}^{AB} g_{AB}. \tag{6}$$

The area 2-form η_{AB} defined on S and corresponding to g_{AB} (and some orientation of S) may also be pulled back to H,

$$\eta_{ab} := \Pi^*{}_{ab}{}^{AB} \eta_{AB}. \tag{7}$$

We use it to define a rotation pseudo scalar Ω,

$$\Omega \eta_{ab} := d\omega_{ab} = \kappa d\tilde{\omega}_{ab}. \tag{8}$$

The Petrov type D equation

The type D equation is imposed on the Riemaniann metric g_{AB} and the rotation pseudo-scalar Ω defined on S. We introduce a complex null co-frame m_A such that the metric g_{AB} and area 2-form η_{AB} read:

$$g_{AB} = m_A \bar{m}_B + m_B \bar{m}_A, \qquad \eta_{AB} = i(\bar{m}_A m_B - \bar{m}_B m_A). \tag{9}$$

The Weyl tensor is of the type D along the generator $\Pi^{-1}(x)$ of the horizon H, if and only if the following equation holds true at point $x \in S$:

$$\bar{m}^A \bar{m}^{B\,(2)}\nabla_A {}^{(2)}\nabla_B \left(K - \tfrac{\Lambda}{3} + i\Omega\right)^{-\frac{1}{3}} = 0, \tag{10}$$

where ${}^{(2)}\nabla_A$ is the torsion free, metric covariant derivative defined by g_{AB}, and the term in the bracket doesn't vanish at x. We solve eq. (10) assuming that the base manifold S (1) is diffeomorphic to a 2-sphere. In that case, all the $U(1)$ bundles are numbered by integers. An integer m corresponding to H can be calculated from the curvature of the $U(1)$-connection 1-form $\tilde{\omega}$, that passes to a condition on the rotation pseudo-invariant Ω

$$\int_{S_2} \Omega \eta_{AB} = 2\pi m \kappa =: 2\pi n. \tag{11}$$

For each Ω there exist 1-forms ω^+ and ω^- defined on S_2 apart from the southern and northern pole respectively such that:

$$d\omega^{\pm}_{AB} = \Omega \eta_{AB}. \tag{12}$$

Coordinates adapted to axial symmetry

We assume that the metric tensor g_{AB} and the rotation pseudo-scalar Ω invariantly defined on S admit an axial symmetry. Consequently, we choose the coordinates adapted to the symmetry in which:

$$g_{AB} dx^A dx^B = R^2 \left(\tfrac{1}{P(x)^2} dx^2 + P(x)^2 d\varphi^2 \right), \tag{13}$$

where $x \in [-1; 1]$, $\varphi \in [0; 2\pi]$ and R is the area parameter. Eq. (10) may be written in terms of Ψ_2-component of the Weyl tensor and in the coordinates adapted to the axial symmetry reads:

$$\partial_x^2 \Psi_2 = 0, \qquad \text{where:} \qquad \Psi_2 = -\tfrac{1}{2}(K + i\Omega) + \tfrac{\Lambda}{6} \tag{14}$$

and its general solution is of the form:

$$\Psi_2 = (c_1 x + c_2)^{-\frac{1}{3}}, \tag{15}$$

where c_1 and c_2 are complex constants. Comparing (15) to the second equality in (14) and expressing Gaussian curvature in the introduced coordinates yields:

$$\frac{1}{(c_1 x + c_2)^3} = \frac{1}{4R^2} \partial_x^2 P^2 - \frac{1}{2} i\Omega + \Lambda'. \tag{16}$$

Solution to the type D equation on the nontrivial bundle topology

The solutions are determined by the cosmological constant Λ, the area radius R^2, a function P and by the rotation pseudo-scalar Ω. The list of $(\Lambda, R^2, P, \Omega)$ we have found is divided into three classes. The first class consists of the metric tensors g_{AB} of constant Gaussian curvature and constant rotation scalar Ω related in the table with n and R^2, and is embeddable in the Taub-NUT-(anti) de Sitter spacetime. The cosmological constant is arbitrary in this class, and unrelated to K and Ω. Hence, that class is parametrized freely by three real parameters Λ', R^2 and n. Class II is characterized by the special relation between R^2 and $\Lambda =: 6\Lambda'$, that is:

$$R^2 = \tfrac{1}{2\Lambda'}, \tag{17}$$

and by the condition

$$\partial_A \Omega \neq 0. \tag{18}$$

The class is parametrized by real parameters Λ', n, α constrained by certain conditions, namely Λ' has to be positive for the area radius R^2 to be positive. It is clear that the frame coefficient P is non-negative for all x in the domain. However, one has to pay attention to the behavior of Ψ_2, eq. (15), on its domain and require it to be well-defined. The third class is the generic one which is parametrized by real parameters $\Lambda', \eta, \gamma, n$ (for parameter constraints see [1]). The issue of embeddability of this class has been discussed in [2].

Possible solutions to the type D equation

Class I	Class II	Class III
$R^2 > 0$	$R^2 = \frac{1}{2\Lambda'}$ and $\Lambda' > 0$	$R^2 = \frac{1}{2}\frac{\gamma}{\Lambda'\gamma - 1} \neq \frac{1}{2\Lambda'}$
$P^2 = 1 - x^2$	$P^2 = 1 - x^2$	$P^2 = \frac{(1-x^2)\left(\left(x - \frac{1}{2}\eta n(1-\Lambda'\gamma)\right)^2 + \eta^2 + \frac{1-x^2}{1-\Lambda'\gamma}\right)}{\left(x - \frac{1}{2}\eta n(1-\Lambda'\gamma)\right)^2 + \eta^2}$
$\Omega = \frac{n}{2R^2}$	$\Omega = -\frac{2\alpha\left(1 - \left(\frac{n\Lambda'}{2\alpha}\right)^2\right)^2}{\left(x - \frac{n\Lambda'}{2\alpha}\right)^3}$	$\Omega = Im\left[\frac{2i\left(1 - \eta^2\left(\frac{1}{2}n(\Lambda'\gamma - 1) + i\right)^2\right)}{\eta\gamma\left(x + \frac{1}{2}\eta n(\Lambda'\gamma - 1) + i\eta\right)^3}\right]$

References and acknowledgments

[1] D. Dobkowski-Ryłko, J. Lewandowski, I. Rácz, **Petrov type D equation on horizons of nontrivial bundle topology**, Phys.Rev.D **100** (2019) 8, 084058.
[2] J. Lewandowski, M. Ossowski, *Kerr-NUT-de Sitter spacetimes*, [arXiv: 2001.10334]

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