Recent Advances on the Hamiltonian Constraint Operator in LQG

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A review on the recent works by
Alesci, Assanioussi, Lewandowski, Mäkinen, Sahlmann, Yang and Ma

Outline

Historic proposals for Hamiltonian constraint operator

- The first idea by [Rovelli, Smolin, 1993]: Constructing the Hamiltonian constraint as a finite operator on the space of diffeomorphism invariant states
- Proposals by [Thiemann 1996]: Rigorous regularization method with graph-dependent triangulation and with flux-holonomy as building blocks

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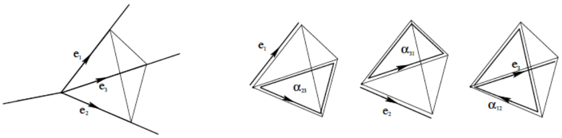
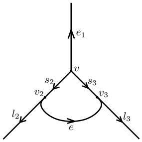
  Scheme 1: Attaching new arcs (edges) and creating new trivalent planar vertices to the spin networks



Historic proposals for Hamiltonian constraint operator

- Scheme 2: Attaching new exceptional edges and creating new trivalent exceptional vertices to the spin networks



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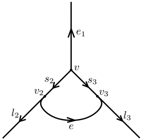
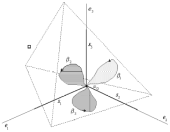


$\longrightarrow$ an anomaly-free (on shell) quantum constraint algebra

- Open problems:
  1. Symmetric Hamiltonian constraint operators?
  2. Ambiguity in averaging different choices of triangulations?
  3. For Scheme 1, degenerate triangulation at the coplanar vertices? For Scheme 2, being less compatible with the spin foam dynamics?

Historic proposals for Hamiltonian constraint operator

- Proposal by [Ashtekar, Lewandowski, 2004]: Attaching new edges, but not creating new vertices to the spin networks



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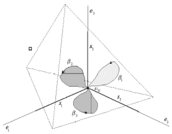
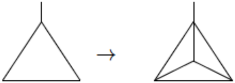


- Open problem: Being disfavored by the spin foam dynamics?
- Proposal by [Alesci, Rovelli, 2010]: Generating 1-4 Pachner moves, more compatible with the spin foam dynamics, creating new 4-valent non-planar vertices to the spin networks



- Open problem: Anomalous quantum constraint algebra?

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  3. Anomaly-free (on-shell) quantum constraint algebra [Thiemann's scheme 2; Ashtekar and Lewandowski]
     $\rightsquigarrow$ The constraint algebra on a habitat [Gambini, Lewandowski, Marolf, Pullin, 1997]
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  4. Correct semiclassical limit
     $\rightsquigarrow$ Issue of long ranged correlation [Smolin 1996]
  5. Being compatible with the spin foam dynamics [Thiemann's scheme 1; Alesci and Rovelli]

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  - Testing the Master Constraint Programme by models [Dittrich, Thiemann, 2004]
  - Constructing master constraint operator for LQG [Thiemann, 2005; Han and Ma, 2005]
  - Correct semiclassical limit in the framework of AQG [Giesel, Thiemann, 2006]
  - Relation to path integral formulation [Han, Thiemann, 2009]

Partially diffeomorphism invariant Hilbert space

- Recall that Thiemann's regulated Euclidean Hamiltonian constraint reads

$$\mathcal{C}^{\epsilon}_{E}(N) \sim \sum_{\Delta\in T(\epsilon)} N(v(\Delta))\epsilon^{ijk} \times \mathrm{Tr}\Big(A(\alpha_{ij}(\Delta))^{-1}A(s_k(\Delta))^{-1}\{A(s_k(\Delta)), V_{R_{v(\Delta)}}\}\Big)$$

- After quantization, the operator cannot keep $\mathcal{H}_{Diff}$ invariant, since the test function N would assign different value $N(v(\Delta))$ for different location of a vertex v of spin network.

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- The creative idea by [Lewandowski, Sahlmann, 2014] is to define certain Hamiltonian constraint operator in a new Hilbert space $\mathcal{H}_{\mathrm{vtx}}$ of partially diffeomorphism invariant states up to vertices of spin networks.

Partially diffeomorphism invariant Hilbert space

- Given a graph γ, one denotes the set of its vertices by $V(\gamma)$, the group of all diffeomorphisms preserving $V(\gamma)$ by $\mathrm{Diff}(\Sigma)_{V(\gamma)}$.
  For any cylindrical function $f_\gamma \in \Phi_{\gamma(V)}$, one can define a map:

$$\eta(f_\gamma) := \frac{1}{N_\gamma} \sum_{\varphi \in \mathrm{Diff}(\Sigma)_{V(\gamma)}/\mathrm{TDiff}(\Sigma)_\gamma} \hat{U}_\varphi \cdot f_\gamma,$$

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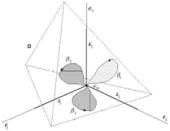
  where $\hat{U}_\varphi$ is the unitary representation of φ, and N_γ is a normalization factor.
- The space $\Phi_{\mathrm{V}} := \eta(\Phi_{\gamma(V)})$ is in the dual space of $\Phi \longrightarrow$ A natural inner product: $\langle \eta(f) | \eta(g) \rangle := \eta(f)(g), \forall f, g \in \Phi$.
- The new Hilbert space is defined as: $\mathcal{H}_{\mathrm{vtx}} := \overline{\bigoplus_{V \in FS(\Sigma)} \mathcal{H}_{\mathrm{V}}}$,
  where $\mathcal{H}_{\mathrm{V}}$ is the completion of Φ_{V}.

Partially diffeomorphism invariant Hilbert space

- In order to have a Hamiltonian constraint operator well defined in $\mathcal{H}_{\mathrm{vtx}}$, the operator can not create new vertices to the spin networks.



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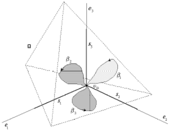

- The regulator can naturally be removed on partially diffeomorphism invariant states.
  $\rightsquigarrow$ A symmetric Hamiltonian constraint operator $\hat{\mathcal{C}}(N) = \sum_{v \in V} N(v)\hat{\mathcal{C}}_v$ well defined in $\mathcal{H}_{\rm vtx}$,
  but disfavored by the spin foam dynamics.

Partially diffeomorphism invariant Hilbert space

- Anomaly-free (on shell) quantum constraint algebra:

$$[\hat{\mathcal{C}}_v,\ \hat{\mathcal{C}}_{v'}] = 0 \ \Rightarrow\ [\hat{\mathcal{C}}(N),\ \hat{\mathcal{C}}(M)] = 0$$

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- Solutions to the quantum constraints
  - Suppose $\hat{\mathcal{C}}_x,\ x \in \Sigma$ are essentially self-adjoint. Then every subspace $\mathcal{H}_{\{x_1,\dots,x_m\}}$ can be decomposed using the spectral decomposition of the operators $\hat{\mathcal{C}}_{x_I},\ I = 1, \dots, m$.
  - The elements of the subspace $\mathcal{H}^{0,\dots,0}_{\{x_1,\dots,x_m\}}$ are solutions to the quantum Hamiltonian constraint as

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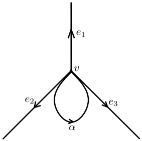
$$\mathcal{H}^{0,\ldots,0}_{\mathrm{V}} \ni \Psi = \sum_{[\gamma]\in[\gamma(V)]} \eta(\Psi_\gamma)$$

  - To turn elements of $\mathcal{H}^{0,\ldots,0}_{\mathrm{V}}$ into solutions to the quantum diffeomorphism constraint, one averages them with respect to the remaining diffeomomorphisms $\mathrm{Diff}(\Sigma)/\mathrm{Diff}(\Sigma)_{V(\gamma)}$.

Special loops and curvature operator

- It was proposed by [Assanioussi, Lewandowski, Mäkinen, 2015] that the Euclidean part of the Hamiltonian constraint operator in $\mathcal{H}_{\mathrm{vtx}}$ should attach "special" loops to the vertices of spin networks, but not create new vertices.



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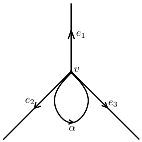

- The Lorentzian part of the Hamiltonian constraint could be replaced by the curvature operator, corresponding to the integral of scalar curvature, constructed in [Alesci, Assanioussi, Lewandowski, 2014].

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- The kernel of the symmetric Hamiltonian constraint operator $\hat{\mathcal{C}}_{\rm sym}(N)$ has the following structure:
  1. Every state that is in the kernel of the volume operator $\hat{V}$ and has coplanar edges at all the vertices of its graph, is in the kernel of $\hat{\mathcal{C}}_{\rm sym}(N)$.
  2. The states of non-zero volume that are in the kernel of $\hat{\mathcal{C}}_{\rm sym}(N)$ have the form of infinite linear combinations of spin network states.
  3. The states of non-zero volume with graphs that do not contain special loops are not in the kernel of $\hat{\mathcal{C}}_{\rm sym}(N)$.

The hint of spin foam models

- Spin foam dynamics [Baez 1997; EPRL-FK 2007]
  $\longrightarrow$ creating new vertices to the spin networks

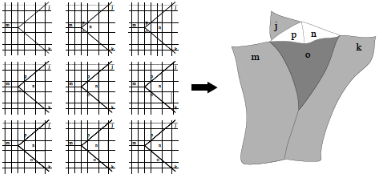

Figure: Perez, 1205.2019

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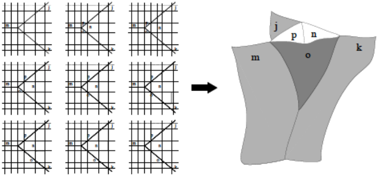

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- Problem: The recent proposed Hamiltonian operators in $\mathcal{H}_{\rm vtx}$ are disfavored by the spin foam dynamics?

Regularization without triangulation

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- By extending an idea discussed in [Thiemann, 2006], the Euclidean Hamiltonian constraint can be regularized as

$$H^E(N) = \frac{1}{2\kappa} \lim_{\epsilon \to 0} \int_\Sigma \mathrm{d}^3x \, N(x) V_{(x,\epsilon)}^{-1/2} \epsilon_{ijk} F^i_{ab}(x) \tilde{E}^a_j(x) \times \int_\Sigma \mathrm{d}^3y \, \chi_\epsilon(x,y) \tilde{E}^b_k(y) V_{(y,\epsilon)}^{-1/2}$$

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- The operator of inverse square root of the volume reads: $\widehat{V_{(y,\epsilon)}^{-1/2}} := \lim_{\lambda\to 0}(\hat{V}_{(y,\epsilon)} + \lambda\ell_{\mathrm{p}}^3)^{-1}\hat{V}_{(y,\epsilon)}^{1/2}$, where $\hat{V}_{(y,\epsilon)}$ is the volume operator in LQG (e.g. [Ashtekar, Lewandowski, 1997])
  $\rightsquigarrow$ Alternative $\widehat{V^{-1}}$ with same feature [Yang, Ma, 2017]

Regularization without triangulation

- We obtain a regulated Hamiltonian operator:

$$\hat{H}^E_\delta(N)\cdot f_\gamma := \frac{\left(\beta\ell_{\mathrm{p}}^2\right)^2}{\kappa\mathcal{N}_\ell}\sum_{v\in V(\gamma)} N_v \widehat{V_v^{-1/2}}$$

$$\times\left(\sum_{e\cap e'=v}\mathrm{sgn}(e',e)\epsilon_{ijk}\mathrm{tr}_\ell\left(h_{\alpha_{e'e}}\tau_i\right)J^j_{e'}J^k_e\right)\widehat{V_v^{-1/2}}\cdot f_\gamma$$

$$=: \sum_{v\in V(\gamma)} N_v \sum_{e\cap e'=v}\hat{H}^E_{v,e'e}\cdot f_\gamma$$

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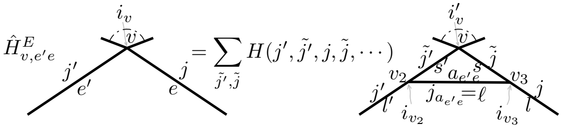
- The assignment of the loop $\alpha_{e'e}$ is diffeormorphism covariant.
- Property of $\widehat{V_v^{-1/2}} \longrightarrow \hat{H}^E_{v,e'e}$ acts nontrivially only on non-planar vertices with valence higher than 3.

Enlarged Partially diffeomorphism invariant states

- The action of $\hat{H}^E_{v,e'e}$ is given by



where $\tilde{j}' \in \{|j'-\ell|,\cdots,j'+\ell\}$,$\tilde{j} \in \{|j-\ell|,\cdots,j+\ell\}$, and $H(j',\tilde{j}',j,\tilde{j},\cdots)$ are coefficients with "$\cdots$" denoting spins associated to other edges incident at v.

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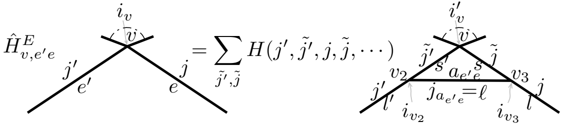

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- The above regularization method can be similarly used to quantize the Lorentzian term in the Hamiltonian constraint.
- To remove the regulator, we consider almost diffeomorphism invariant states, by averaging over the images of spin network states under diffeomorphisms, but leaving fixed sets of non-planar vertices with valence higher than 3 invariant.

Enlarged Partially diffeomorphism invariant states

- Given a graph γ, we denote its non-planar vertices with valence higher than 3 by $V_{\rm np4}(\gamma)$, the group of all diffeomorphisms preserving $V_{\rm np4}(\gamma)$ by $\mathrm{Diff}(\Sigma)_{V_{\rm np4}(\gamma)}$. For any cylindrical function $f_\gamma \in \Phi$, we define a map by

$$\eta(f_\gamma) := \frac{1}{N_\gamma} \sum_{\varphi \in \mathrm{Diff}(\Sigma)_{V_{\rm np4}(\gamma)}/\mathrm{TDiff}(\Sigma)_\gamma} \hat{U}_\varphi \cdot f_\gamma,$$

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  where $\hat{U}_\varphi$ is the unitary representation of φ, and N_γ is a normalization factor.
- The space $\Phi_{\rm np4} := \eta(\Phi)$ is in the dual space of $\Phi \longrightarrow$ A natural inner product: $\langle \eta(f)|\eta(g)\rangle := \eta(f)(g), \forall f, g \in \Phi$.
- The new Hilbert space $\mathcal{H}_{\rm np4}$ is defined as the completion of $\Phi_{\rm np4}$, along the way similar to that of $\mathcal{H}_{\rm vtx}$ in [Lewandowski and Sahlmann].

Symmetric Hamiltonian constraint operator

- We can use the freedom of choosing the spin representations ℓ [Gaul, Rovelli, 2000] attached to each new added loop to ensure that the valence of a vertex would not be changed by the action of $\hat{H}_\delta(N) \longrightarrow$

$$\left((\hat{H}(N))' \cdot \phi\right)(f_\gamma) := \lim_{\delta\to 0} \phi(\hat{H}_\delta(N) \cdot f_\gamma), \ \forall f \in \Phi, \ \phi \in \Phi_{\mathrm{np4}}$$

- A corresponding symmetric Hamiltonian constraint operator:

$$(\hat{H}_{\mathrm{sym}}(N))' := \frac{1}{2}\left((\hat{H}(N))' + ((\hat{H}(N))')^\dagger\right)$$

is well defined in $\mathcal{H}_{\mathrm{np4}}$.

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is well defined in $\mathcal{H}_{\rm np4}$.

  - $(\hat{H}(N))'$ will annihilate the arcs created by $\hat{H}_\delta(N)$.
  - $((\hat{H}(N))')^\dagger$ will create planar trivalent vertices.

    $\rightsquigarrow$ The algebra of $(\hat{H}_{\rm sym}(N))'$ is anomaly-free on shell.

    $\rightsquigarrow$ No need to average different choices of triangulations.

    $\rightsquigarrow$ Being compatible with the spin foam dynamics.

Possible approach to semiclassical analysis

- Rewrite the standard spin network states in terms of representation matrices and intertwiners projected onto the basis of angular momentum coherent states [Alesci, Lewandowski, Mäkinen, 2016; Talk of Mäkinen].

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  - Let $|\iota> \in Inv(\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N})$ be an intertwiner in a standard recoupling theory basis, $|j\xi>$ be the coherent states of angular momentum parametrized in terms of a complex number ξ. The projection of the intertwiner onto the coherent state reads

$$\iota_{\xi_1 \cdots \xi_N} :=< \iota|\, j_1 \xi_1 \cdots j_N \xi_N > .$$

  - From the explicit expression of $\iota_{\xi_1 \cdots \xi_N}$ one can extract a representation of $|\iota>$ as a polynomial of $\xi_1 \cdots \xi_N$.

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  - From the explicit expression of $\iota_{\xi_1 \cdots \xi_N}$ one can extract a representation of $|\iota >$ as a polynomial of $\xi_1 \cdots \xi_N$.
  - The operators including the Hamiltonian can be reformulated as differential operators acting on these polynomials.
    $\rightsquigarrow$ Opening up a way towards investigating the semiclassical limit of the dynamics via asymptotic approximation methods.

Physical Hamiltonian by deparametrization

- In the deparametrized model of LQG coupled to a free scalar field, one can construct a physical Hamiltonian operator in $\mathcal{H}_{\rm vtx}$ [Alesci, Assanioussi, Lewandowski, Mäkinen, 2015]
  - The idea of so-called special loops and curvature operator are suitable for this construction.
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Physical Hamiltonian by deparametrization

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  - An approximation method was developed by [Assanioussi, Lewandowski, Mäkinen, 2017] to deal with the physical Hamiltonian operators [Talk by Assanioussi].
- Full quantization of gravity coupled to Klein-Gordon scalar field [Lewandowski, Sahlmann, 2015]
  - A non-standard representation of the scalar field was employed.
  - A new Hamiltonian constraint operator $\hat{C}(N)$ was defined on certain partially diffeomorphism invariant states $\eta(<\varphi|\otimes<\gamma,j,\iota|)\longrightarrow$

$$[\hat{C}(M),\hat{C}(N)]\sim\sum_{e\in\gamma}\sqrt{j_e(j_e+1)}\int_e sgn(d\varphi)(NdM-MdN).$$

Summary

- Recent progress on constructing Hamiltonian constraint:
  i New Hilbert spaces $\mathcal{H}_{\rm vtx}$ or $\mathcal{H}_{\rm np4}$ of (enlarged) diffeomorphism invariant states up to vertices (or non-planar vertices with valence higher than 3) were proposed as the domain of the Hamiltonian constraint operators.
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    $\longrightarrow$ Creating arcs and new trivalent co-planar vertices to the spin networks.
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- The approach to find the solutions of the quantum constraints was proposed for the Hamiltonian in $\mathcal{H}_{\mathrm{vtx}}$, which can be naturally generalized to $\mathcal{H}_{\mathrm{np4}}$.

Summary

- Realized properties of the quantum Hamiltonian:
  - ✓ Being symmetric and well-defined in a suitable Hilbert space
  - ✓ Anomaly-free (on-shell) quantum constraint algebra
  - ✓ Avoiding the triangulation ambiguity
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  - ✓ Systemical approach to solve the quantum constraints.
- Recent advances related to the quantum Hamiltonian:
  - A new way was suggested towards the semiclassical analysis of the Hamiltonian by rewriting spin network states in terms of angular momentum coherent states.
  - Physical Hamiltonian operators in $\mathcal{H}_{\rm vtx}$ were proposed in deparametrized models, and an approximation method was developed for actual calculation.
  - Full quantum treatments for gravity coupled to a scalar field were developed, with interesting findings.
  - Issues of off-shell constraint algebra was discussed in models [Varadarajan, 2016; Lewandowski, Lin, 2016].

Outlook

- Calculating the matrix elements of the Hamiltonian constraint operators
  $\rightsquigarrow$ e.g., a cross checking of the matrix elements of Thiemann's Hamiltonian constraint with trivalent vertices by graphical calculation [Yang, Ma, 2017].
- Extending the deparametrized treatments for $\mathcal{H}_{\mathrm{vtx}}$ to $\mathcal{H}_{\mathrm{np4}}$.

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  $\rightsquigarrow$ e.g., a cross checking of the matrix elements of Thiemann's Hamiltonian constraint with trivalent vertices by graphical calculation [Yang, Ma, 2017].
- Extending the deparametrized treatments for $\mathcal{H}_{\mathrm{vtx}}$ to $\mathcal{H}_{\mathrm{np4}}$.
- Testing the new proposed Hamiltonian operators by symmetry-reduced models
  $\rightsquigarrow$ The new regularization technique can be applied to LQC [Yang, Ma, 2015], and a new Hamiltonian constraint operator was obtained:

$$\hat{H} = -\frac{3}{\kappa\beta^2}\widehat{V^{-1/2}}\frac{\sin^2(\bar{\mu}c)}{\bar{\mu}^2}\hat{p}\hat{p}\widehat{V^{-1/2}}, \quad \bar{\mu}^2|p| = 2\sqrt{3}\pi\beta\ell_{\mathrm{p}}^2.$$

!Thanks!