Problems on General Relativity: 9

December 11, 2021

Problem 1. Consider the Schwarzschild metric tensor

$$g = -(1-\frac{2M}{r})dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

for $r > r_s$ and a timelike geodesic $\gamma(\tau) = (t(\tau), r, \frac{\pi}{2}, \phi(\tau))$, where τ is the proper time parameter of a particle moving along the curve. Calculate:

1. the length of the orbit as a function of r and M.
2. the total energy relative to a static observer at infinity per unit mass, that is

   $$E := -\dot{\gamma}_a T^a, \qquad T := \partial_t, \tag{2}$$

   as a function of r and M.
3. the proper time $\Delta\tau$ of a segment of the curve as a function of r, M and Δt (the time of a distant static observer)
4. the time for one lap
5. the value of E at the last stable orbit.
6. the value of E at the last (in a sense) unstable orbit
7. remember "Interstellar"? ...

Problem 2. In the same spacetime consider a null geodesic $\gamma(\tau) = (t(\tau), r(\tau), \frac{\pi}{2}, \phi(\tau))$.

1. Derive the equation

   $$\frac{1}{2}\dot{r}^2 + \frac{L^2}{2r^3}(r-2M) = \frac{1}{2}E^2, \qquad L := \dot{\gamma}_a \Phi^a, \qquad \Phi := \partial_\phi. \tag{3}$$

2. find the maximum r_{ph} of V, and its value $V(r_{\mathrm{ph}})$. Does r_{ph} correspond to the ray of light going around the center of the space in a circle?
3. calculate the minimum value of the apparent impact parameter $\frac{E}{L}$ required to surmount the top of the effective potential.
4. derive $\frac{d\phi}{dr}$ given M and $\frac{E}{L}$.

Problem 3* optional. Suppose that from a distance the $r = 2M$ sphere looks like a black ball against a brighter sky. How does its appearance change as we approach it. Why does the sky at a certain distance become almost a dot behind the back of the person looking towards the sphere?