Problems on General Relativity: 7

November 27, 2021

Below in the Problem 1 - 3 we consider an n-dimensional manifold M equipped with a metric tensor g. ∇ is the corresponding covariant derivative (metric and torsion free), $R^a{}_{bcd}$, R_{bc} and R is the Riemann tensor, Ricci tensor and Ricci scalar, respectively. Except for Problem 2, g is assumed to satisfy the Einstein equations (vacuum, that is without matter)

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 0.$$

Problem 1. Show that: the Riemann tensor satisfies

$$\nabla_a R^a{}_{bcd} = 0 \tag{1}$$

Problem 2. Show that if $n = 2$, then

$$R_{ab} - \frac{1}{2}R_{ab} = 0 \tag{2}$$

for every g.

Problem 3. Show, that if $n = 3$, then the Riemann tensor is isotropic at every point (invariant with respect to the transformations of the tangent space preserving g), and vanishes in the case $\Lambda = 0$.

Hint:
Step (i): assume that the Weyl tensor is identically zero.
Step (ii): Using the symmetries of the Riemann tensor, prove that the Weyl tensor vanishes at every point of M.

Problem 4* optional. Consider 4-dimensional manifold M, and the Palatini Lagrangean

$$L(e, \omega) := \eta_{abcd} e^a \wedge e^b \wedge \Omega^{cd} \tag{3}$$

where the variables are differential 1-forms e^a and $\omega^a{}_b$, $(a, b = 0, 1, 2, 3)$, such that for every pair (e, ω), the $\omega^a{}_b$s set a metric connection (no condition on the torsion) of the metric tensor g, defined as follows

$$g := -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 =: \gamma_{ab} e^a \otimes e^b, \tag{4}$$

η is the volume tensor, namely

$$\eta = e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \frac{1}{4!}\eta_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \tag{5}$$

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{6}$$

Calculate the Noether current $j^{(\xi)}$ corresponding to a vector field ξ in M.
Hint:

$$j^{(\xi)} = \theta(\mathcal{L}_\xi \omega) - \xi \lrcorner L$$

where θ is defined by

$$\delta L = E_a \delta e^a + E_a{}^b \delta\omega^a{}_b + d\theta(\delta\omega).$$