Problems on General Relativity: 6

November 18, 2021

Problem 1. Suppose a metric tensor g admits a time-like Killing vector field ξ, that is

$$\mathcal{L}_\xi g = 0, \qquad \xi^a \xi_a < 0. \tag{1}$$

Consider a family o stationary observers of the velocities

$$u^a = \frac{\xi^a}{\sqrt{-\xi_b \xi^b}}. \tag{2}$$

Show that the acceleration

$$a^a := \nabla_u u^a \tag{3}$$

is

$$a^a = \frac{1}{2} \frac{g^{ab} \nabla_b (\xi^c \xi_c)}{\xi^d \xi_d} \tag{4}$$

Practice that formula on the following example

$$-N(r) dt \otimes dt + \frac{1}{N(r)} dr \otimes dr + g_{AB}(r, x^1, x^2) dx^A \otimes dx^B, \qquad A, B = 1, 2, \qquad \xi = \partial_t \tag{5}$$

Helpful tip: $\xi^a \xi_a$ is constant along the integral curves of ξ.

Problem 2* - optional. Consider a metric tensor g of signature $-+...+$, that is such that can be written as

$$g = -e^0 \otimes e^0 + e^1 \otimes e^1 + ... + e^{n-1} \otimes e^{n-1}. \tag{6}$$

A co-dimension 1 surface N is said to be null, if it is orthogonal to a non vanishing vector field n such that

$$n_a n^a|_N = 0. \tag{7}$$

Show that every null surface is woven by null geodesics.
Helpful tip: The vector field n can be chosen to be

$$n_a = \nabla_a f$$

where f is a function.