Problems on General Relativity: 4

October 28, 2021

Problem 1. Consider a cylinder $\mathbb{R} \times S_3$ embedded in 5-dimensional Minkowski spacetime $\mathbb{M}_5$,

$$\Phi \;:\; \mathbb{R} \times S_3 \to \mathbb{M}_5. \tag{1}$$

The Minkowski spacetime is endowed with a metric tensor

$$g = -(dZ^0)^2 + (dZ^1)^2 + ... + (dZ^4)^2, \tag{2}$$

where $(Z^0, ..., Z^4)$ are coordinates. The embedding is defined as follows

$$\Phi^0(t, \chi, \theta, \varphi) \equiv Z^0(t, \chi, \theta, \varphi) = a \sinh t \tag{3}$$

$$\Phi^1(t, \chi, \theta, \varphi) \equiv Z^1(t, \chi, \theta, \varphi) = a \cosh t \cos \chi \tag{4}$$

$$\Phi^2(t, \chi, \theta, \varphi) \equiv Z^2(t, \chi, \theta, \varphi) = a \cosh t \sin \chi \cos \theta \tag{5}$$

$$\Phi^3(t, \chi, \theta, \varphi) \equiv Z^3(t, \chi, \theta, \varphi) = a \cosh t \sin \chi \sin \theta \cos \varphi \tag{6}$$

$$\Phi^4(t, \chi, \theta, \varphi) \equiv Z^4(t, \chi, \theta, \varphi) = a \cosh t \sin \chi \sin \theta \sin \varphi. \tag{7}$$

(ii) Derive the pull back $\Phi^* g$.

Problem 2. Consider $\mathbb{R}^4$ embedded in 5-dimensional Minkowski spacetime $\mathbb{M}_5$,

$$\Phi \;:\; \mathbb{R}^4 \to \mathbb{M}_5 \tag{8}$$

introduced in Problem 1. The embedding is defined as follows

$$\Phi^0(\eta, x, y, z) \equiv Z^0(\eta, x, y, z) = \frac{a^2 + s}{2\eta} \tag{9}$$

$$\Phi^1(\eta, x, y, z) \equiv Z^1(\eta, x, y, z) = \frac{a^2 - s}{2\eta} \tag{10}$$

$$\Phi^2(\eta, x, y, z) \equiv Z^2(\eta, x, y, z) = \frac{ax}{\eta} \tag{11}$$

$$\Phi^3(\eta, x, y, z) \equiv Z^3(\eta, x, y, z) = \frac{ay}{\eta} \tag{12}$$

$$\Phi^4(\eta, x, y, z) \equiv Z^4(\eta, x, y, z) = \frac{az}{\eta} \tag{13}$$

$$s := -\eta^2 + x^2 + y^2 + z^2 \tag{14}$$

(i) Show, that the image of Φ is contained in the hyperboloid

$$-(Z^0)^2 + (Z^1)^2 + (Z^2)^2 + (Z^3)^2 + (Z^4)^2 = a^2 \tag{15}$$

(ii) Derive the pull back $\Phi^* g$.

Problem 3* optional. The isometries of $(\mathbb{M}_5, g)$ are believed to be the flows of the linear combinations of the following 15 vector fields

$$\xi = \partial_{Z^0}, \partial_{Z^1}, \partial_{Z^2}, \partial_{Z^3}, \partial_{Z^4}, \tag{16}$$

$$Z^1 \partial_{Z^0} + Z^0 \partial_{Z^1}, ..., Z^4 \partial_{Z^0} + Z^0 \partial_{Z^4}$$

$$Z^1 \partial_{Z^2} - Z^2 \partial_{Z^1}, ..., Z^4 \partial_{Z^3} - Z^3 \partial_{Z^4}$$

(i) Show, that indeed

$$\mathcal{L}_\xi g = 0.$$

(ii) Find those among the vectors (16) which are tangent to the hyperboloid (15).
(iii) What is a conclusion on the dimension of the isometry group of the hyperboloid (15)?