Problems on General Relativity: 3

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Problem 1. Let $\vec{r}$ and r denote the following vector field and function

$$\vec{r} = x^i \partial_i \quad \text{and} \quad r = \sqrt{x^2 + y^2 + z^2} \tag{1}$$

respectively, defined in $\mathbb{R}^3$.

Consider a metric tensor

$$q = dx \otimes dx + dy \otimes dy + dz \otimes dz \tag{2}$$

and the vector field:

$$X = 2x\vec{r} - r^2 \partial_x \tag{3}$$

called *inverted translation*.

Show, that

$$\mathcal{L}_X q = 4xq \tag{4}$$

where $\mathcal{L}_X$ stands for the Lie derivative with respect to X.

Recall, that the general formula for the Lie derivative of a rank-2 tensor is

$$\mathcal{L}_X(q_{ij} dx^i \otimes dx^j) = X(q_{ij}) dx^i \otimes dx^j + q_{ij} d(X(x^i)) \otimes dx^j + q_{ij} dx^i \otimes d(X(x^j)) \tag{5}$$

Problem 2*- optional. Consider the metric tensor of round sphere

$$q = d\theta^2 + \sin^2\theta d\varphi^2. \tag{6}$$

Find a vector field B parallel to ∂_θ, that is

$$B = f(\theta)\partial_\theta \tag{7}$$

where f is to be found, such that there is a function h such that

$$\mathcal{L}_B q = hq. \tag{8}$$

Problem 3*- optional. Consider de Sitter metric tensor

$$g = \frac{r_c^2}{\eta^2}(-d\eta \otimes d\eta + dx \otimes dx + dy \otimes dy + dz \otimes dz), \tag{9}$$

and the vector field

$$T = \eta\partial_\eta + x\partial_x + y\partial_y + z\partial_z. \tag{10}$$

Show, that

$$\mathcal{L}_T g = 0. \tag{11}$$