Problems on General Relativity: 2

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Problem 1. Consider the following vector field

$$v = x\partial_t + t\partial_x \tag{1}$$

defined in $\mathbb{R}^2$.

(i) For every point (t_0, x_0) find a curve

$$[-\tau_1, \tau_1] \ni \tau \mapsto (t(\tau), x(\tau)) \tag{2}$$

such that

$$(t(0), x(0)) = (t_0, x_0), \tag{3}$$

and

$$\frac{d}{d\tau}\left(t\left(\tau\right), x\left(\tau\right)\right) = v_{|(t(\tau),x(\tau))} \tag{4}$$

for every τ.

(ii) For every value of τ, find

$$\Phi_\tau : \mathbb{R}^2 \to \mathbb{R}^2 \tag{5}$$

such that

$\frac{d}{d\tau}\left(\Phi_\tau\left(t,x\right)\right) = v_{|\Phi_\tau(t,x)}$

Problem 2. In the subset $\{(t,x) : x > 0\} \subset \mathbb{R}^2$ introduce coordinates (ρ, τ) related to the standard (t,x) in the following way

$$t = \rho\,\mathrm{sh}(\tau), \qquad x = \rho\,\mathrm{ch}(\tau) \tag{6}$$

where

$$\mathrm{sh}(\tau) := \frac{e^\tau - e^{-\tau}}{2}, \qquad \mathrm{ch}(\tau) := \frac{e^\tau + e^{-\tau}}{2} \tag{7}$$

Express the vector fields $(\partial_\tau, \partial_\rho)$ in terms of the coordinates (t,x).

Problem 3. On S_3 parametrised by the Euler angles (ψ, θ, ϕ) an important role is played by the following vector fileds

$$\begin{aligned}
X_1 &= \cos\psi\partial_\theta + \frac{\sin\psi}{\sin\theta}\partial_\phi - \frac{\sin\psi}{\tan\theta}\partial_\psi,\\
X_2 &= -\sin\psi\partial_\theta + \frac{\cos\psi}{\sin\theta}\partial_\phi - \frac{\cos\psi}{\tan\theta}\partial_\psi,\\
X_3 &= \partial_\psi,\\
Y_1 &= \cos\phi\partial_\theta - \frac{\sin\phi}{\tan\theta}\partial_\phi + \frac{\sin\phi}{\sin\theta}\partial_\psi,\\
Y_2 &= -\sin\phi\partial_\theta - \frac{\cos\phi}{\tan\theta}\partial_\phi + \frac{\cos\phi}{\sin\theta}\partial_\psi,\\
Y_3 &= \partial_\phi.
\end{aligned} \tag{8}$$

(i) Calculate the commutators $[X_i, X_j], i,j = 1,2,3$.

(ii) Show, that the 1-forms

$$\begin{aligned}
\omega^1 &= \cos\psi d\theta + \sin\theta\sin\psi d\phi,\\
\omega^2 &= -\sin\psi d\theta + \sin\theta\cos\psi d\phi,\\
\omega^3 &= d\psi + \cos\theta d\phi,
\end{aligned} \tag{9}$$

set a basis dual to (X_1, X_2, X_3).

($iii*$) (for enthusiasts only) Calculate the commutators $[X_i, Y_j]$, and $[Y_i, Y_j]$ for $i,j = 1,2,3$. Find the basis dual to (Y_1, Y_2, Y_3).