Problems on General Relativity: 11

December 29, 2021

Problem 1. Consider the Kottler spacetime ("Schwarzschild–de Sitter")

$$g_{\mathrm{K}} = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

$$f(r) = 1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2, \tag{2}$$

$$r \geq 0, \quad \Lambda,\, M \geq 0, \quad 0 \leq 9\Lambda M^2 \leq 1.$$

1. Calculate the non-zero root r_{dS} of f, in the case $M = 0$.
2. Calculate the double root r_{e} of f, in the extremal mass case, that is when $9\Lambda M^2 = 1$.
3. Let $r_{\mathrm{b}} < r_{\mathrm{c}}$ be the two positive roots of f, in the case $0 < 9\Lambda M^2 < 1$; put the numbers: $r_{\mathrm{S}} := 2M, r_{\mathrm{b}}, r_{\mathrm{c}}, r_{\mathrm{e}}, r_{\mathrm{dS}}$ in growing order.
4. For what values of the variable r is the spacetime (locally) static?

Problem 2. Consider an observer at rest in the Kottler spacetime (that is, on the world line tangent to the Killing vector field ∂_t).

1. Calculate the acceleration vector a of the observer,

$$a := \nabla_u u, \tag{3}$$

where u is the observer 4-velocity.

2. Calculate the magnitude of the acceleration

$$|a| = \sqrt{a_\mu a^\mu}.$$

3. What is the limit of $|a|$ when r approaches r_b or r_c?
4. For what value $r = r_{\mathrm{f}}$ does the observer move freely along a geodesic?
5. For what values of r is the observer attracted inward by spacetime, and for what values is the observer repelled outward?
6. Can you place the value r_{f} between the values of r considered in Problem 1.3?

Problem 3. Consider null geodesics

$$\gamma(\tau) = (t(\tau), r(\tau), \theta = \frac{\pi}{2}, \phi(\tau)) \tag{4}$$

and derive the potential $V(r; L)$ such that

$$\frac{1}{2}\dot{r}^2 + V(r; L) = \frac{1}{2}E^2, \tag{5}$$

where L is the angular momentum. Which properties of the null geodesics depend on the value of Λ and which do not?

Problem 4. Describe the black hole horizon $r = r_{\mathbf{b}}$ using the generalised Eddington-Finkelstein coordinates (v, r, θ, ϕ) defined by

$$dv := dt + \frac{dr}{f}, \tag{6}$$

and the cosmological horizon $r = r_{\mathbf{c}}$ using coordinates (u, r, θ, ϕ) such that

$$dv := dt - \frac{dr}{f}. \tag{7}$$

Extend the Killing vector

$$\xi = \partial_t \tag{8}$$

to the horizons, and find the values of the surface gravities κ_{b}, κ_{c}. Recall, that

$$\nabla_\xi \xi|_{r=r_{\mathbf{b}}} = \kappa_{\mathrm{b}} \xi, \qquad \nabla_\xi \xi|_{r=r_{\mathbf{c}}} = \kappa_{\mathrm{c}} \xi. \tag{9}$$