Problems on General Relativity: 10

December 19, 2021

Problem. The Schwarzschild spacetime

$$g_{\mathrm{S}} = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$
$$f(r) = 1 - \frac{2M}{r}$$

in the reference system of a family of free-falling observers.

1. Find a vector field X that can be expressed by a function τ in the following way,
$$X^a\partial_a = -g_{\mathrm{S}}{}^{ab}\tau_{,b}\partial_a \tag{2}$$
where τ satisfies
$$g_{\mathrm{S}}{}^{ab}\tau_{,a}\tau_{,b} = -1. \tag{3}$$
Suggested form of τ:
$$d\tau = dt + T(r)dr, \tag{4}$$
derive $T(r)$.

2. Notice, that the integral lines of X describe a family of freely moving point observers. Choose $T(r)$ such that they are falling on the center.

3. Find a co-vector
$$d\rho = dt + \sigma(r)dr \tag{5}$$
such that
$$X^a\partial_a\rho = 0, \tag{6}$$
derive $\sigma(r)$.

4. Write the vector field X in the coordinate system $(\tau, \rho, \theta, \phi)$.

5. Write the metric tensor (1) in the coordinate system $(\tau, \rho, \theta, \phi)$. At that stage, consider the old coordinate r as a given function
$$r = r(\tau, \rho) \tag{7}$$
and show, that
$$g_{\mathrm{S}} = -d\tau^2 + F(r)d\rho^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2); \tag{8}$$
calculate the function $F(r)$.

6. Derive the relation (explicit or, if impossible, implicit) $r(\tau, \rho)$.

7. Conclude, that
$$F(r)d\rho^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{9}$$
is the instantaneous geometry (dependent on τ) of the 3-dimensional observer space.