Problems on General Relativity: 1

October 14, 2021

Problem 1. Consider two stereographic projections,

$$\mathbb{R}^3 \supset \{p = (p^x, p^y, p^z) : (p^x)^2 + (p^y)^2 + (p^z)^2 = 1\} \to \mathbb{R}^2$$

$$\chi(p) = (x(p), y(p)) = (\frac{p^x}{1-p^z}, \frac{p^y}{1-p^z}) \tag{1}$$

$$\chi'(p) = (x'(p), y'(p)) = (\frac{p^x}{1+p^z}, \frac{p^y}{1+p^z}). \tag{2}$$

Derive the following map,

$$\chi \circ \chi'^{-1} \tag{3}$$

and specify its domain.

Does the map (3) preserve an orientation? Can you improve it?

Problem 2. Consider a 4-dimensional vector space V and it's dual V^*. Let (e^1, e^2, e^3, e^4) be a basis of V^*. In the space of the rank 2 antisymmetric tensors consider the following product

$$(E|F) = \frac{1}{4} E_{ab} F_{cd} \epsilon^{abcd} \tag{4}$$

where ϵ^{abcd} is the totally antisymmetric symbol such that

$$\epsilon^{1234} = 1, \tag{5}$$

and

$$E = \frac{1}{2} E_{ab} e^a \wedge e^b, \qquad F = \frac{1}{2} F_{ab} e^a \wedge e^b \tag{6}$$

Show, that:

$$E \wedge F = (E|F) e^1 \wedge e^2 \wedge e^3 \wedge e^4 \tag{7}$$

and

$$(E|F) = (F|E). \tag{8}$$

In what way the product depends on the choice of the basis (e^1, e^2, e^3, e^4)?

Find tensors E^1, E^2, E^3 and F^1, F^2, F^3, such that

$$(E^1|E^1) = (E^2|E^2) = (E^3|E^3) = 2 = -(F^1|F^1) = -(F^2|F^2) = -(F^3|F^3) \tag{9}$$

while all the other products vanish.

Clue: You may start with

$$E^1 = e^1 \wedge e^2 + e^3 \wedge e^4, \qquad F^1 = e^1 \wedge e^2 - e^3 \wedge e^4 \tag{10}$$