Exam on General Relativity

February 5, 2021

In the problems $1-12$ consider vacuum spacetime around spherically symmetric black hole at the presence of a positive cosmological constant Λ, characterised by the Kottler metric tensor:

$$g_{\mathrm{SdS}} = -(1-\frac{r_S}{r}-\frac{r^2}{\ell^2})dt^2 + \frac{dr^2}{1-\frac{r_S}{r}-\frac{r^2}{\ell^2}} + r^2\left(d\theta^2+\sin^2\theta d\phi^2\right), \qquad \ell := \sqrt{\frac{3}{\Lambda}}. \tag{1}$$

1. What values r_b and r_c of the coordinate r correspond to black hole, and cosmological horizon, respectively? **Can you find exact explicit expressions? (It is not necessary to solve the remaining problems). For what values of r the vector field ∂_r is timelike?
2. Arrange the four values r_b, r_c, r_S and ℓ in ascending order.
3. Calculate acceleration of an observer in rest (that is, such that $r, \theta, \phi =$const). Is the acceleration directed toward or outward the black hole horizon?
4. Find a value $r = r_f$ such that the acceleration of the corresponding observer vanishes. What does it mean for the observer's world line?
5. An observer in rest at $r = r_1$ emits a ray of light to an observer at rest at $r_2 > r_1$. What is the relationship between light frequencies according to observers? Does the second observer see the light "red shifted" or "blue shifted"?
6. Find an extension of the spacetime that contains the black hole horizon to the future of the region $r_b < r < \ell$. In order to do it, replace the coordinate t by a suitable function u.

   Hint: consider
   $$du := dt \pm \frac{dr}{1-\frac{r_S}{r}-\frac{\ell^2}{r^2}}$$
   and carefully choose either + or −.
7. Find an extension of the spacetime that contains the cosmological horizon to the future of the region $r_b < r < \ell$. In order to do it, replace the coordinate t by a suitable function v.

   Hint: consider
   $$dw := dt \pm \frac{dr}{1-\frac{r_S}{r}-\frac{\ell^2}{r^2}}$$
   and carefully choose either − or +.
8. * Show that, the function r decreases / increases for every curve future directed, in the region $r < r_b$ / $r > \ell$ of the extension defined in 5 / 6.

   Hint: notice, that in the coordinates $(u/v, r, \theta, \phi)$, the vector field ∂_r is null, and carefully selected factor of $\pm$ makes it future directed, consider the sign of $\frac{dx^\mu}{d\tau}(\partial_r)_\mu$.
9. * Consider the extensions of the vector field ∂_t to the black hole horizon and the cosmological horizon, respectively, of item 5 and 6. Calculate the corresponding surface gravities.
10. Consider a timelike or null geodesic $\tau \mapsto (t(\tau), r(\tau), \theta_0 = \frac{\pi}{2}, \phi_0)$. Derive effective potential $V(r; J, \Lambda)$ such that
    $$\frac{1}{2}(\frac{dr}{d\tau})^2 + V(r; J, \Lambda) = \frac{1}{2}E^2 \quad \text{where} \quad J = \frac{dx^\mu}{d\tau}(\partial_\phi)_\mu, \quad E = -\frac{dx^\mu}{d\tau}(\partial_t)_\mu \tag{2}$$
11. Find unstable circular orbit for a point particle moving at speed of light.

12. ** Assuming that $\ell >> r_S$ and J is given, derive radius of circular orbit to the first leading order in $\frac{r_S}{\ell}$. Find the minimal stable circular orbit, and the minimal unstable circular orbit, in that approximation.

    Hint: if $R(\Lambda, J)$ is the exact value of radius depending on Λ, that is

    $$\partial_r V(R(\Lambda, J), J; \Lambda) = 0$$

    then

    $$\frac{d}{d\Lambda}\partial_r V(R(\Lambda, J), J; \Lambda) = 0.$$

13. Consider spacetime

    $$g = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

    Calculate the Kretschmann scalar

    $$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$$

    and derive a condition on $a(t)$ at t_0 such that

    $$a(t_0) = 0,$$

    that ensures the scalar is still finite.

14. Consider a timelike geodesic

    $$\tau \mapsto (x^\mu(\tau)) = (t(\tau), x(\tau), 0, 0)$$

    parametrised by proper time τ. Derive $t(\tau)$ and $x(\tau)$ assuming $a(t)$ is given as well as $t(\tau_0), x(\tau_0)$.

    Hint: What about $\frac{dx^\mu}{d\tau}(\partial_x)_\mu$?

15. Consider spacetime endowed with the metric tensor

    $$-d\tau^2 + a^2(\tau)(d\psi^2 + \mathrm{sh}^2\psi(d\theta^2 + \sin^2\theta d\phi))$$

    Solve The Friedmann equations for

    $$\rho = -P.$$

    Hint: See Lecture 15

16. *In the previous problem, consider the family of comoving observers x, y, z =const. Who of them can communicate with the observer $(x, y, z) = (0, 0, 0)$ in the period $\tau \in (0, \tau_0)$.

17. ** Prove that for every metric tensor g in 4-dimensional spacetime, locally there exist coordinates (t, x, y, z) such that

    $$g = -dt^2 + g_{ab}(t, x, y, z)dx^a dx^b, \qquad x^a = x, y, z.$$

    Hint: Show that

    $$(\nabla_X X = 0, \;\; g(X, X) = -1, \;\; \mathcal{L}_X Y = 0) \;\Rightarrow\; X^\mu\partial_\mu(g(X, Y)) = 0.$$