Problems on General Relativity: 6

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Problem 1. A bird chases a fly near the horizon of a black hole. In an instant the fly crosses the horizon and flies to the other side. A little later, the bird does the same. What does a bird see before, during, and after crossing the horizon, assuming it is closely watching the fly? Will the fly sink into the horizon and disappear for a while? Will the bird have to find the fly with its eyes on the other side later? Draw bird and fly paths and light rays to illustrate your answer. Use the qualitative properties of spacetime viewed in the advanced Eddington-Finkelstein coordinates that penetrate the black hole horizon:

$$g_{\text{Sch}} \;=\; -(1-\frac{r_s}{r})du^2 + 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

Problem 2. Consider the original Schwarzshild spacetime

$$g_{\text{Sch}} = -(1-\frac{r_S}{r})dt^2 + \frac{dr^2}{1-\frac{r_S}{r}} + \left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad r > r_s, \tag{2}$$

and assume that the vector field

$$\partial_t$$

is oriented to the future.

Next, introduce the advanced Eddington-Finkelstein coordinates defined by

$$dt = du - \frac{dr}{1-\frac{r_s}{r}}$$

that turn the metric tensor into (1) and extend it allowing all the positive values of r including $r = r_s$,

$$\infty < u < \infty, \quad 0 < r.$$

i) transform the vector field ∂_t to the Eddington-Finkelstein coordinates system
ii) find two linearly independent everywhere null, future directed vector fields
iii) show, that at every point such that

$$r < r_s$$

every timelike future directed vector field X satisfies

$$X^r < 0$$

iv) consider a timelike geodesic

$$(x^\mu(\tau)) = (u(\tau), r(\tau), \theta = \pi/2, \phi(\tau)), \quad \dot{x}^\mu \dot{x}_\mu = -1,$$

contained in the interior region, that is such that

$$r < r_s,$$

and by using the method of the constants of motion derive $\dot{r}$ in the form

$$\dot{r} = f(r; J, E)$$

where

$$J := \dot{x}^\mu(\partial_\phi)_\mu, \qquad E := -\dot{x}^\mu(\partial_u)_\mu.$$

Problem 3. The aim of this problem is to describe geometrically the following report:

The astronaut is slowly approaching the horizon of the black hole. At first she sees a star-flecked sky in front of him and a black hole in the middle. As the astronaut approaches, the black hole grows to such an extent that the sky and stars appear now behind him. As the astronaut moves even closer to the horizon, the sky and stars form a bright disc that tapers behind him. The disk narrows to a point when the astronaut reaches the horizon.

To that end, consider an observer in rest at an integral curve of the Killing vector field ∂_t in the spacetime (2) contained in the exterior region

$$r > r_s.$$

Without lack of generality assume that the observer's $\theta = \frac{\pi}{2}$, and consider incoming geodesics

$$(x^\mu(\tau)) = (t(\tau), r(\tau), \theta = \pi/2, \phi(\tau)), \qquad \dot{x}^\mu \dot{x}_\mu = 0.$$

Apply the equation

$$\dot{r}^2 + \frac{J^2(r - r_s)}{r^3} = E^2, \qquad J = \dot{\phi} r^2$$

to the light rays (that is tangent vectors to the null geodesics) reaching the observer.

Importantly, every ray incoming to the observer is characterised by its optical angle ψ defined geometrically as follows

$$\dot{r}\partial_r + \dot{\phi}\partial_\phi = \cos\psi n_r + \sin\psi n_\phi, \qquad n_r = \sqrt{1 - \frac{r_s}{r}}\partial_r, \;\; n_\phi = \frac{1}{r}\partial_\phi$$

The optical size of the horizon the astronaut can see, is the maximal value 2ψ takes in the set of all the incoming null geodesics such that at some point $x^\mu(\tau_0)$ in the past

$$r(\tau_0) = r_s.$$