Problems on General Relativity: 3

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Problem 1. Consider two stereographic projections,

$$\mathbb{R}^3 \supset \{(x', y', z') : x'^2 + y'^2 + z'^2 = 1\} \to \mathbb{R}^2$$

$(x', y', z') \mapsto (x, y)$ and $(x', y', z') \mapsto (x", y")$ defined as follows:

$$(x, y) = (\frac{x'}{1 - z'}, \frac{y'}{1 - z'}) \tag{1}$$

$$(x", y") = (\frac{x'}{1 + z'}, \frac{y'}{1 + z'}). \tag{2}$$

Derive the map

$$\psi : (x", y") \mapsto (x, y) \tag{3}$$

and specify its domain.

Problem 2. Using the map (3) calculate the pullback

$$g" := \psi^* g$$

of the metric tensor

$$g = \frac{4}{(1 + x^2 + y^2)^2}(dx^2 + dy^2)$$

obtained by the pull back of the flat metric in $\mathbb{R}^3$ with the inverse map $\mathbb{R}^2 \to \mathbb{R}^3$ to (1). Can you explain the result?

Problem 3. Consider a general $1-$form $\omega_x(x, y)dx + \omega_y(x, y)dx$ and its pullback $\omega" := \psi^*\omega$. Formulate necessary and sufficient conditions satisfied by the functions ω_x, ω_y, for the functions $\omega"_{x"}(x", y")$ and $\omega"_{y"}(x", y")$ to be well defined at $(x", y") = (0, 0)$.