Problems on General Relativity: 2

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Problem 1. Consider the following co-vector (1-form) fields

$$\begin{aligned} e^1 &= \cos\psi d\theta + \sin\theta \sin\psi d\phi \\ e^2 &= -\sin\psi d\theta + \sin\theta \cos\psi d\phi \\ e^3 &= d\psi + \cos\theta d\phi \end{aligned} \tag{1}$$

defined in a coordinate system (ψ, θ, ϕ) valid in $[0, 2\pi] \times [0, \pi] \times [0, 2\pi] \subset \mathbb{R}^3$. At every point (ψ, θ, ϕ) find tangent vectors e_1, e_2, e_3, such that (e^1, e^2, e^3) is dual to (e_1, e_2, e_3)

Problem 2. For the vector fields (e_1, e_2, e_3) derived in Problem 1, calculate the commutators and write the result in the following form

$$[e_i, e_j] = c_{ij}{}^k e_k \tag{2}$$

Problem 3. Knowing that given a vector field V, the Lie derivative $\mathcal{L}_V$ satisfies

$$\mathcal{L}_V\,(f dh)) = V(f)dh + f d(V(h)), \qquad \mathcal{L}_V(\omega + \omega') = \mathcal{L}_V\omega + \mathcal{L}_V\omega' \tag{3}$$

for every pair of functions f, h and co-vector fields ω, ω', calculate $\mathcal{L}_{e_i} e^j$ for the vector fields e_i and co-vector fields e^j defined in Problem 1.

Problem 4. For each of the co-vector fields (e^1, e^2, e^3) derived in Problem 1, calculate the exterior derivative and write the result in the following form

$$de^i = c^i{}_{jk} e^j \wedge e^k. \tag{4}$$

Compare the result with Problem 2.