Problems on General Relativity 1

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A family of observers is sitting in Minkowski spacetime on a ring of a radius R (according to an inertial observer at rest with respect to the center of the ring) rotating at angular velocity ω. Find:

- the proper time of each observer as a function of the inertial time
- the proper circumference of the ring according to the observers
- the discrepancy in synchronising the observers clocks

Clarification:

In the inertial coordinates (x^0, x^1, x^2, x^3) an observer sitting at a point ϕ of the ring has the following world line

$$(x^0, x^1, x^2, x^3) = (x^0, R\cos(\phi + \frac{\omega}{c}x^0), R\sin(\phi + \frac{\omega}{c}x^0), 0) \tag{1}$$

Assume, there is one observer per each $\phi \in [0, 2\pi)$.

Clue:

Introduce the rotating coordinate system (x^0, r, ϕ, x^3) where r and ϕ are such that:

$$x^1 = r\cos(\phi + \frac{\omega}{c}x^0), \quad x^2 = r\sin(\phi + \frac{\omega}{c}x^0) \tag{2}$$

Describe the world lines of the observers in the rotating coordinates. Find a simultaneity line set by observers. The metric tensor written in terms of the rotating coordinates is:

$$-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \;=\; -(1 - \frac{\omega^2 r^2}{c^2})(dx^0 - \frac{r^2\omega/c}{1 - \frac{\omega^2 r^2}{c^2}}d\phi)^2 + dr^2 + \frac{r^2}{1 - \frac{\omega^2 r^2}{c^2}}d\phi^2 + dx^3 \tag{3}$$